PARASTATISTICS AS EXAMPLES OF THE EXTENDED HALDANE STATISTICS

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Abstract

We show that for every algebra of creation and annihilation operators with a Fock-like representation, one can define extended Haldane statistical parameters in a unique way. Specially for parastatistics, we calculate extended Haldane parameters and discuss the corresponding partition functions.

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Introduction.- There are two distinct approaches to generalized statistics: (i) the first approach obeys some symmetry principle (permutation group, braid group, quantum groups and algebras, etc.) (ii) the second approach is defined by counting the number of independent multiparticle quantum states .

The first approach can be characterized by an operator algebra of creation and annihilation operators with a Fock-like representation (for example, parastatistics[1,2], infinite quon statistics [3], anyons[4], etc.).Parastatistics[1,2] were the first consistent generalization of Bose and Fermi statistics in any spacetime dimension. The N-(para)particle states can occur in nontrivial representations of the permutation group S_N, i.e. not only in totally symmetric (for bosons) or totally antisymmetric (for fermions).

The second approach is characterized by some Hilbert space of quantum states, generally without a direct connection with creation and annihilation operators acting on Fock-like space. This class includes the recently suggested Haldane generalization of the Pauli exclusion principle, interpolating between Bose and Fermi statistics . The Haldane statistics g of a particle is defined by [5]

$$g = \frac{d_n - d_{n+\Delta n}}{\Delta n}, \tag{1}$$

where n is the number of particles and d_n is the dimension of the one-particle Hilbert space obtained by keeping the quantum numbers of $(n - 1)$ particles fixed. For bosons, $g = 0$ and for fermions, the Pauli principle implies $g = 1$.

Although Haldane statistics were proposed in any spacetime dimension, all examples known so far (Calogero-Sutherland model[6] , 1D spinons[5] with $g = \frac{1}{2}$, anyons residing in the lowest Landau level in a strong magnetic field [7]) are essentially

1D systems. Furthermore, there is no clear connection between Haldane exclusion statistics and the operator algebra.In particular, Green's parastatistics [1] which also generalize the Pauli exclusion principle have not been described as a type of Haldane statistics. It is natural to ask the question:is it possible to define the *extended* Haldane statistics interpolating between para-Bose and para-Fermi statistics as Haldane statistics interpolates between the Bose and Fermi statistics.

In this Letter we show that any operator algebra of creation and annihilation operators with a Fock-like representation can be described in terms of the *extended* Haldane statistics parameters. We apply it to the para-Bose and para-Fermi algebras [1,2] of M oscillators and calculate few lowest extended Haldane parameters for illustration. They describe parastatistics as the original Haldane parameters $g = 0$ (1) describe Bose (Fermi) statistics. We also discuss partition functions for free systems and the counting of independent multiparticle states.

Extended Haldane statistics parameters.- Let us start with any algebra of M pairs of creation and annihilation operators $a_i^\dagger$, a_i, $i = 1, 2, ..M$ ($a_i^\dagger$ is Hermitian conjugated to a_i). The algebra is defined by a normally ordered expansion Γ (generally no symmetry principle is assumed)

$$a_i a_j^\dagger = \Gamma_{ij}(a^\dagger; a), \tag{2}$$

with the number operators N_i, i.e., $[N_i, a_j^\dagger] = a_i^\dagger \delta_{ij}$, $[N_i, a_j] = -a_i \delta_{ij}$.We assume[8] that there is a unique vacuum $|0>$ and the corresponding Fock-like representation. The scalar product is uniquely defined by $<0|0>= 1$, the vacuum condition $a_i|0>= 0$, $i = 1, 2, ..M$, and Eq.(2).A general N-particle state is a linear combination of the vectors $(a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0>)$, $i_1, \cdots i_N = 1, 2, ...M$. We consider Fock spaces

with no state vectors of negative squared norms.Note that we do not specify any relation between the creation (or annihilation) operators themselves. They appear implicitly as norm zero vectors in Fock space. For a state $(a^{\dagger}_{i_1}\cdots a^{\dagger}_{i_N}|0>)$ with fixed indices $i_1,\cdots i_N$ we write $1^{n_1}2^{n_2}...M^{n_M}$, where $n_1, n_2...n_M$ are eigenvalues of the number operators $N_1,\cdots N_M$, satisfying $\sum_{i=1}^{M} n_i = N$. Then there are $\frac{N!}{n_1!n_2!...n_M!}$ (in principle different) states obtained by permutations $\pi \in S_N$ acting on the state $(a^{\dagger}_{i_1}\cdots a^{\dagger}_{i_N}|0>)$. From these vectors we form a Hermitian matrix $\mathcal{A}(i_1,\cdots i_N)$ of their scalar products.The number of linearly independent states among them is given by $d_{i_1,\cdots i_N} = rank\mathcal{A}(i_1,\cdots i_N)$.

The set of $d_{i_1,\cdots i_N}$ for all possible $i_1,\cdots i_N = 1,2,\cdots M$ and all integers N completely characterizes the statistics and the thermodynamic properties of a *free* system with the corresponding Fock space.Note that the statistics,i.e., the set $d_{i_1,\cdots i_N}$ do not uniquely determine the algebra given by Eq.(2). Following Haldane's idea [5] , we define the dimension of the one-particle subspace keeping the $(N-1)$ quantum numbers $i_1,\cdots i_{N-1}$ inside the N-particle states, fixed :

$$d^{(1)}_{i_1,\cdots i_{N-1}} = \sum_{j=1}^{M} d_{j,i_1,\cdots i_{N-1}}. \tag{3}$$

For M Bose oscillators, $d_{i_1,\cdots i_N} = 1$, $d^{(1)}_{i_1,\cdots i_{N-1}} = M$ for any choice $i_1,\cdots i_N$. For M Fermi oscillators, $d_{i_1,\cdots i_N} = 1$ and $d^{(1)}_{i_1,\cdots i_{N-1}} = M-(N-1)$ if $i_1,\cdots i_N$ are mutually different (otherwise $d_{i_1,\cdots i_N} = d^{(1)}_{i_1,\cdots i_{N-1}} = 0$). We point out that $d_{i_1,\cdots i_N}$ and $d^{(1)}_{i_1,\cdots i_{N-1}}$ are integers, i.e., no fractional dimension is allowed by definition.

Recall that Haldane introduced statistics parameter g through the change of the single-particle Hilbert space dimension d_n ,Eq.(1). In the similar way we define *extended* Haldane statistics parameters $g_{i_1,\cdots i_{N-1};j_1\cdots j_k}$ through the change of available

one-particle Fock-subspace dimension $d^{(1)}_{i_1,\cdots i_{N-1}}$, Eq.(3),i.e.

$$g_{i_1,\cdots i_{N-1};j_1\cdots j_k} = \frac{d^{(1)}_{i_1,\cdots i_{N-1}} - d^{(1)}_{i_1,\cdots i_{N-1};j_1\cdots j_k}}{k}. \tag{4}$$

Note that Eq.(4) implies that extended Haldane statistics parameters can be any rational number (see also discussion after Eq.(15)). For bosons, $g_{i_1,\cdots i_{N-1};j_1\cdots j_k} = 0$, $\forall i_1,\cdots i_{N-1}, j_1\cdots j_k$, and for fermions, $g_{i_1,\cdots i_{N-1};j_1\cdots j_k} = 1$ if $i_1,\cdots i_{N-1}, j_1\cdots j_k$ are mutually different. Note that the quantities $g_{i_1,\cdots i_{N-1},i_N,j}$, obtained by adding only one particle to the N-particle system, cannot be fractional by definition. This indicates that for the original Haldane statistics with fractional g , ($g = 1/p$, $p \in \mathbf{N}$), Eq.(1), there is no underlying operator algebra of creation and annihilation operators with Fock-like representations. It seems that the original Haldane statistics cannot be realized in the above sense, except for free (or weakly interacting) bosons and fermions (see also discussion in Ref.(5)).

The number of all independent N-particle states distributed over M quantum states described by M independent oscillators ($i = 1, 2\cdots M$) is given by

$$D(M,N;\Gamma) = \sum_{i_1,\cdots i_N=1}^{M} d_{i_1,\cdots i_N}. \tag{5}$$

Note that $0 \le D(M,N) \le M^N$ and $D(M,N)$ is always an integer by definition. For M Bose oscillators, $D^B(M,N) = \begin{pmatrix} M+N-1 \\ N \end{pmatrix}$ and for M Fermi oscillators, $D^F(M,N) = \begin{pmatrix} M \\ N \end{pmatrix}$.

Wu [9] suggested a simple interpolation between the Bose and the Fermi counting

$$D(M,N;g) = \frac{[M+(N-1)(1-g)!]}{N![M-gN-(1-g)]!} \tag{6}$$

with $g = 0$ corresponding to bosons and $g = 1$ corresponding to fermions. Equation (6) was used in Refs.(9,10) to obtain the partition function and the thermo-

dynamic properties (in 1D and 2D systems). Karabali and Nair [11] started with the counting rule given by Eq.(6) and, using a few additional assumptions, derived an operator algebra defined basically by $(\sum_{i=1}^{M} c_i a_i)^{p+1} = 0$, $p = \frac{1}{g} \in \mathbf{N}$.This operator algebra interpolates between the Bose ($p = \infty$) and the Fermi ($p = 1$) algebra. However, the counting rule (5), calculated for the Karabali-Nair algebra,is $D(M,N;p) = D^B(M,N)$ if $N \leq p$; $D(M,N;p) = 0$ if $N > Mp$, and $1 \leq D(M,N;p) < D^B(M,N)$ if $p < N \leq Mp$, which is obviously different from Eq.(6).Moreover, the corresponding partition function for the system with the free Hamiltonian $H_0 = \sum_{i=1}^{M} E_i N_i$ is $\mathcal{Z}(M;p) = \prod_{i=1}^{M} \frac{x_i^{p+1}-1}{x_i-1}$, $x_i = e^{-\frac{E_i}{kT}}$, and leads to the thermodynamic properties different from those obtained in Ref.(9).An ad hoc defined counting formula for $D(M,N)$ can lead in a plausible (statistical) way to the partition function and the corresponding thermodynamic properties. However, the underlying operator algebra may not exist.

Parastatistics.- The para-Bose and para-Fermi operator algebras are defined by trilinear relations

$$[a_i^{\dagger} a_j \pm a_j a_i^{\dagger}, a_k^{\dagger}] = (\frac{2}{p})\delta_{jk} a_i^{\dagger}, \quad \forall i,j,k = 1,2,..M. \tag{7}$$

The upper (lower) sign corresponds to the para-Bose (para-Fermi) algebra,and p is the order of parastatistics.The unique vacuum $|0>$ is assumed with the conditions $<0|0>= 1$, $a_j|0>= 0$, $a_i a_j^{\dagger}|0>= \delta_{ij}|0>$, $i,j = 1,...M.$. The corresponding Fock space does not contain any state with negative squared norm only if p is an integer or $p = \infty$.[1,2] Consistency, i.e. the structure of null-states requires

$$[a_i^{\dagger}, [a_j^{\dagger}, a_k^{\dagger}]_{\pm}] = 0, \qquad \forall i,j,k. \tag{8}$$

The trilinear relations(7) can be presented in the form of Eq.(2) [8]. Using relations (7) and (8), one finds by induction [2,8]

$$\begin{aligned} a_i a^{\dagger}_{i_1} \cdots a^{\dagger}_{i_N}|0> &= \sum_{k=1}^{N} \delta_{ii_k} \epsilon^{k-1} \, a^{\dagger}_{i_1} \cdots \hat{a}^{\dagger}_{i_k} \cdots a^{\dagger}_{i_N}|0> \\ &- (\frac{2}{p}) \sum_{k=2}^{N} \delta_{ii_k} \sum_{l=1}^{k-1} \epsilon^{l} a^{\dagger}_{i_1} \cdots \hat{a}^{\dagger}_{i_l} \cdots a^{\dagger}_{i_{k-1}} a^{\dagger}_{i_l} a^{\dagger}_{i_{k+1}} \cdots a^{\dagger}_{i_N}|0>, \end{aligned} \tag{9}$$

where $\epsilon = \mp$,the upper (lower) sign is for parabosons (parafermions) and the sign " $\wedge$ " denotes omission of the corresponding operator. The matrices $\mathcal{A}(i_1, \cdots i_N)$ can be calculated recursively using Eq.(9). Since Eqs.(7)-(9) are invariant under any choice of indices $(1, 2, \cdots M)$, the matrix $\mathcal{A}(i_1, \cdots i_N)$ does not depend on any particular choice of $(i_1, \cdots i_N)$, but it depends only on the corresponding partition λ of N ($N = \sum_{a=1}^{M} \lambda_a = |\lambda|$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_M \geq 0$), i.e.,the Young tableau of the permutation group S_N. Hence, we write $\mathcal{A}(i_1, \cdots i_N) \equiv \mathcal{A}_{\lambda}$.
If the indices $(i_1, \cdots i_N)$ are mutually different, the corresponding partition is denoted by 1^N,with the corresponding $N! \times N!$ *generic* matrix $\mathcal{A}_{1^N}$.All other matrices $\mathcal{A}_{\lambda}$, $|\lambda| = N$ are easily obtained from a generic matrix $\mathcal{A}_{1^N}$, given by

$$\mathcal{A}_{1^N} = \sum_{\pi \in S_N} f(\pi)\, R(\pi)\,. \tag{10}$$

Here R is the right regular representation of the permutation group S_N and $f(\pi)$ is a real function depending on $\epsilon = \mp 1$ and the order of parastatistics p.For p integer there is no state with negative squared norm. However, there are null-states, so that $0 \leq d_{1^N} \leq N!$.The Hermitian matrix $\mathcal{A}_{1^N}$ commutes with every permutation σ in the left regular representation. Hence, the nondegenerate quotient Fock space splits

into the sum of irreducible representations (IRREP's) of S_N, and we can write

$$d_{1^N} = \sum_{\mu} n(\mu)\, K_{\mu,1^N} \qquad n(\mu) \geq 0, \tag{11}$$

where the sum runs over all partitions μ of N, and $K_{\mu,1^N}$ are Kostka's numbers [12], i.e., the dimension of the irreducible representation (IRREP) μ of the group S_N, and $n(\mu)$ is an integer denoting the multiplicity of IRREP μ, which can be determined from the spectrum of the matrix $\mathcal{A}_{1^N}$.

For parastatistics, the function $f(\pi)$, $\pi \in S_N$, in Eq.(11), is given up to $N = 4$ by

$$\begin{aligned}
f(1234) &= 1, \\
f(2134) &= f(1324) = f(1243) = \epsilon q, \\
f(2314) &= f(3124) = f(2143) = f(1423) = f(1342) = q^2, \\
f(3214) &= f(1432) = \epsilon Q, \\
f(2413) &= f(3142) = f(4123) = f(2341) = \epsilon q^3, \\
f(3421) &= f(4231) = f(4312) = \epsilon q(1 - 2q + 2Q), \\
f(3241) &= f(4132) = f(4213) = f(2431) = qQ, \\
f(3412) &= 1 - 2q^2 + 2qQ, \\
f(4321) &= q(2Q - q),
\end{aligned} \tag{12}$$

where $q = 1 - \frac{2}{p}$ and $Q = q^2 + q - 1$.

Our general result for d_λ, where λ is a partition of N, $|\lambda| = N \leq 4$, obtained from the 24×24 matrix $\mathcal{A}_{1^4}$, is given by

$$d_\lambda = rank[\mathcal{A}_\lambda] = \sum_{\mu;|\mu|=|\lambda|} n(\mu)\, K_{\mu,\lambda}\,, \tag{13}$$

where, for parabosons, $n(\mu)^{pB} = 1$ if $l(\mu) \leq M$ and $l(\mu) \leq p$ (otherwise $n(\mu)^{pB} = 0$); for parafermions, $n(\mu)^{pF} = 1$ if $l(\mu) \leq M$ and $l(\mu^T) \leq p$ (otherwise $n(\mu)^{pF} = 0$).The $l(\mu)$ and $l(\mu^T)$ are the number of rows of the Young tableaux μ and μ^T, respectively, and μ^T denotes the transposed tableau to μ. Kostka's number $K_{\mu,\lambda}$ is the filling number of an IRREP μ with independent states arranged according to the partition λ, $|\mu| = |\lambda|$. Hence, from all allowed equivalent IRREP's μ, only one appears in the decomposition (13). It is proved[13] that the pattern for the multiplicities $n(\mu)$ is valid for any M,N,λ ,p . Specially,for $M = 2$, $\forall\lambda, N, p$, and for $p = 2$, $\forall\lambda$, N, M we reproduce the results of Refs.(14,15). Generally, the thermodynamic properties of a free system are completely defined by the set of all $d_\lambda \neq 0$, or by the set of all $n(\mu)$.

Let us write the one-particle dimension $d^{(1)}_{i_1,\cdots i_{N-1}} \equiv d^{(1)}_\lambda$ given by Eq.(3). If the partition λ of $(N-1)$ is given by $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_j > 0$, $(j \leq M)$, where $\lambda_1, \cdots \lambda_j$ are multiplicities of identical quantum numbers among $i_1, \cdots i_{N-1}$, then

$$d^{(1)}_\lambda = d_{(\lambda_1+1,\lambda_2,\cdots\lambda_j)} + d_{(\lambda_1,\lambda_2+1,\cdots\lambda_j)} + \cdots + d_{(\lambda_1,\lambda_2,\cdots\lambda_j+1)} + (M-j)d_{(\lambda_1,\lambda_2,\cdots\lambda_j,1)}. \quad (14)$$

For example, $d^{(1)}_0 = M$, $d^{(1)}_1 = d_2 + (M-1)d_{(1,1)}$, $d^{(1)}_2 = d_3 + (M-1)d_{(2,1)}$, $d^{(1)}_{(1,1)} = 2d_{(2,1)} + (M-2)d_{(1,1,1)}$, etc. Then, the parameters of the extended Haldane statistics [Eq.(4)] are

$$g_{\lambda\to\mu} = \frac{d^{(1)}_\lambda - d^{(1)}_\mu}{|\mu| - |\lambda|}, \qquad \lambda \subset \mu. \quad (15)$$

If $d^{(1)}_\mu = 0$, the transition $\lambda \to \mu$, $\lambda \subset \mu$, is forbidden. A natural definition of exclusion statistics is as follows: for every λ there exists μ , $\lambda \subset \mu$ and $l(\lambda) = l(\mu)$ such that $d_\mu = 0$. Para-Fermi statistics of order $p \in \mathbf{N}$ are examples of exclusion statistics since at most p particles can occupy a given quantum state, $a_i^{p+1} = 0$.

For a single oscillator $a^{p+1} = 0$, the extended statistical parameters are $g_{i\to j} = 0$ for $j \leq p$ and $g_{i\to p+1} = \frac{1}{p+1-i}$. Para-Bose statistics are not examples of exclusion statistics (except for $p = \infty$).

We present the extended Haldane parameters $g_{\lambda\to\mu}$ [Eq.(15)] for some special cases.

For parabosons:

g	$0 \to 1$
p=1	0
p$\geq$ 2	-(M-1)

g	$1 \to 2$	$1 \to 1^2$
p=1	0	0
p=2	0	-(M-1)
p$\geq$ 3	0	-(2M-3)

For parafermions:

g	$0 \to 1$
p=1	1
p$\geq$ 2	-(M+1)

g	$1 \to 2$	$1 \to 1^2$
p=1	M-1	1
p=2	M	-(M-3)
p$\geq$ 3	0	-(2M-3)

Finally, let us write the partition functions for free parastatistical systems with the Hamiltonian $H_0 = \sum_{i=1}^{M} E_i\, N_i$, where N_i is the number operator for the i^{th} quantum state and E_i is the corresponding one-particle energy. Then the partition function $\mathcal{Z}_N$ for N-particle states is

$$\mathcal{Z}_N(x_1, ...x_M; p, \epsilon) = \sum_{\lambda;|\lambda|=N} d_\lambda(p, \epsilon) m_\lambda(x_1, ...x_M)\,, \qquad x_i = e^{-\frac{E_i}{kT}}, \tag{16}$$

where d_λ is given by Eq.(13) and $m_\lambda(x_1, ...x_M)$ denotes the monomial S_M-symmetric function[12] corresponding to the partition λ of N , namely, $m_\lambda(x_1, ...x_M) =$

$= \sum_{\pi \in S_M} x_1^{\lambda_1} \cdots x_M^{\lambda_M}$, where the sum is over all distinct permutations of $(\lambda_1, \cdots \lambda_M)$. Using the results (13) we can write

$$\mathcal{Z}_N(x_1, ...x_M; p, \epsilon) = \sum_{\mu; |\mu|=N} S_\mu(x_1, ...x_M), \tag{17}$$

where $S_\mu(x_1, ...x_M)$ are the Schur S_M-invariant functions [12], corresponding to the IRREP μ of S_M. The sum in Eq.(17) is restricted to $l(\mu) \leq M$ and $l(\mu) \leq p$ for parabosons, i.e. to $l(\mu) \leq M$ and $l(\mu^T) \leq p$ for parafermions. Note also that the counting formula (5) is $D(M, N; p, \epsilon) = \mathcal{Z}_N(\underbrace{1, 1, ...1}_{M}; p, \epsilon)$. If $N \leq p$, $\mathcal{Z}_N^{pB} = \mathcal{Z}_N^{pF}$ and if $N > p$, $\mathcal{Z}_N^{pB} > \mathcal{Z}_N^{pF}$.

The partition function for M para-Bose oscillators with $p \geq M$ is simple (and generalizes the result for $M = 2$ of Ref.(15)):

$$\mathcal{Z}^{pB}(x_1, ...x_M; p \geq M) = \sum_\lambda S_\lambda(x_1, ...x_M) = \prod_{i=1}^{M} \frac{1}{(1 - x_i)} \prod_{i<j}^{M} \frac{1}{(1 - x_i x_j)}. \tag{18}$$

The general results for any S_M-invariant algebra (2) are $\mathcal{Z}_N(x_1, ...x_M; \Gamma) = \sum_\mu n(\mu) S_\mu(x_1, ...x_M)$, where $n(\mu)$ is the multiplicity of IRREP μ in the decomposition (11). The thermodynamic properties of such systems will be treated separately.

Note added. In recent preprints [16,17],which partly overlap with our paper, the authors follow the first quantized approach to parastatistics.In this Letter we follow Green's second quantized approach.

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References

1. H.S.Green, Phys.Rev.**90** (1953) 170;O.W.Greenberg and A.M.L.Messiah, Phys.Rev.B **138** (1965) 1155;J.Math.Phys. **6** (1965) 500; Y.Ohnuki and S.Kamefuchi, *Quantum Field Theory and Parastatistics* (University of Tokio Press, Tokio, Springer, Berlin, 1982).

2. A.B.Govorkov, Theor.Math.Phys.**98** (1994) 107.

3. O.W.Greenberg, Phys. Rev.D **43** (1991) 4111; R.N.Mohapatra, Phys.Lett.B, **242** (1990) 407 ; S.Meljanac and A.Perica, Mod.Phys.Lett.A **9** (1994) 3293.

4. J.M.Leinaas and J.Myrheim, Nuovo Cim. **37** (1977) 1 ; F.Wilczek, Phys.Rev.Lett. **48** (1984) 1144.

5. F.D.M.Haldane, Phys.Rev.Lett. **67** (1991) 937.

6. S.Isakov, Int.J.Mod.Phys.A. **9** (1994) 2563; Mod.Phys.Lett.B **8** (1994) 319; M.V.N.Murthy and R.Shankar, Phys.Rev.Lett. **73** (1994) 331;see also Comment by A.Dasnieres de Veigy and S.Ouvry, Phys.Rev.Lett. **75** (1994) 331.

7. A.Dasnieres de Veigy and S.Ouvry, Phys.Rev.Lett. **72** (1994) 600; Mod.Phys.Lett.A **10** (1995) 1.

8. S.Meljanac and M.Milekovic , Int.J.Mod.Phys.A. **11** (1996) 1391.

9. Y.S.Wu, Phys.Rev.Lett. **73** (1994) 922.

10. C.Nayak and F.Wilczek, Phys.Rev.Lett. **73** (1994) 2740; S.Isakov et al., *Thermodynamics for Fractional Statistics* , (preprint cond-mat/9601108).

11. D.Karabali and V.P.Nair, Nucl.Phys.B **438** **[FS]** (1995) 551.

12. I.G.Macdonald, *Symmetric Functions and Hall Polynomials* (Claredon, Oxford, 1979).

13. S.Meljanac,M.Stojic and D.Svrtan, *Partition functions for general multilevel systems*,(preprint hep-th/9605064).

14. P.Suranyi, Phys.Rev.Lett. **65** (1990) 2329.

15. A.Bhattacharyya,F.Mansouri,C.Vaz and L.C.R.Wijewardhana, Phys.Lett.B, **224** (1989) 384;Mod.Phys.Lett.A **12** (1989) 1121 .

16. S.Chaturvedi, *Canonical partition functions for parastatistical system of any order*, (preprint hep-th/9509150).

17. A.Polychronakos, *Path integrals and parastatistics*,(preprint hep-th/9603179).